

Fuchsian systems for Dotsenko–Fateev multipoint correlation functions and similar integrals of hypergeometric type

Abstract

The Dotsenko–Fateev integral is an analytic function of one complex variable expressing the amplitude in the 4-point correlator of the 2D conformal field theory. Dotsenko–Fateev found ODE of third order with Fuchsian singularities satisfied by their integral. In the present paper, this work is extended to generalized Dotsenko–Fateev integrals, in particular those associated to arbitrary multipoint correlators, and Pfaff systems of PDE of Fuchsian type are constructed for them. The ubiquity of the Fuchsian systems is in that they permit to obtain local expansions of solutions in the neighborhoods of singularities of the system.

Keywords: Dotsenko–Fateev integral, systems of partial differential equations, Pfaffian systems of Fuchsian type, hyperplane arrangements, hypergeometric functions

1. Introduction

In two-dimensional conformal field theory, a $(n + 2)$ -point correlation function containing a third order conformal operator can be expressed, using the projective invariance to fix arbitrary three points, in terms of special correlators of the following form:

$$\begin{aligned} & \langle \phi_{\alpha_{k_0, l_0}}(0) \phi_{\alpha_{k_1, l_1}}(1) \phi_{\alpha_{1, 3}}(x_2) \dots \phi_{\alpha_{1, 3}}(x_n) \phi_{\alpha_{k_3, l_3}}(\infty) \rangle = \\ & \int_{C_1 \times C_2} \langle V_{\alpha_{k_1, l_1}}(0) V_{\alpha_{k_2, l_2}}(1) V_{\alpha_{1, 3}}(x_2) \dots V_{\alpha_{1, 3}}(x_n) V_{\alpha_{k_3, l_3}}(\infty) V_{\alpha_+}(t) V_{\alpha_+}(s) \rangle dt ds \sim \\ & \prod_{i=2}^n x_i^{2\alpha_{k_1, l_1} \alpha_{1, 3}} \prod_{i=2}^n (1 - x_i)^{2\alpha_{k_2, l_2} \alpha_{1, 3}} \int_{C_1 \times C_2} t^{2\alpha_{k_1, l_1} \alpha_+} s^{2\alpha_{k_1, l_1} \alpha_+} (t - 1)^{2\alpha_{k_2, l_2} \alpha_+} (s - 1)^{2\alpha_{k_2, l_2} \alpha_+} \times \\ & \prod_{i=2}^n (t - x_i)^{2\alpha_{1, 3} \alpha_+} (s - x_i)^{2\alpha_{1, 3} \alpha_+} (t - s)^{2\alpha_+^2} dt ds, \end{aligned} \quad (1.1)$$

where $\alpha_{k, l} = \frac{1-k}{n} \alpha_- + \frac{1-l}{n} \alpha_+$ are quantum charges, α_{\pm} are two roots of the equation $\Delta_{\alpha_{\pm}} = \alpha_{\pm}^2 - 2\alpha_{\pm} \alpha_0 = 0$ (i.e. the conformal dimension of the integral operator $\int_{C_i} V_{\alpha_{\pm}}(t) dt$ should be zero; α_0 is some constant), V_{α} is the Coulomb operator with charge α (i.e. the exponent $e^{i\alpha\phi}$ of the free field ϕ), C_i are the integration paths satisfying the conditions $\mathring{C}_i \subset \bar{\mathbb{C}} \setminus \{0, 1, x_2, \dots, x_n, \infty\}$, $\partial C_i \in \{0, 1, x_2, \dots, x_n, \infty\}$. The integral contained in the above expression for the case $n = 2$ is called Dotsenko–Fateev integral. It was investigated in [1, 2], where an ordinary differential equation of third order was found, of which it is a solution.

The present paper is devoted to the investigation of the integral defining the correlation function (1.1) for arbitrary n , that is the integral of the following form:

$$I_n(a_0, a_1, \dots, a_n, g; x_2, \dots, x_n) = \int_{C_1 \times C_2} t^{a_0} s^{a_0} (t - 1)^{a_1} (s - 1)^{a_1} \prod_{i=2}^n (t - x_i)^{a_i} (s - x_i)^{a_i} (t - s)^g dt ds, \quad (1.2)$$

where $a_i \in \mathbb{C}$ are complex parameters, and C_j are integration paths satisfying the conditions $\mathring{C}_j \subset \bar{\mathbb{C}} \setminus \{0, 1, x_2, \dots, x_n, \infty\}$, $\partial C_j \in \{0, 1, x_2, \dots, x_n, \infty\}$. Moreover, the natural generalization of the Dotsenko–Fateev integral

$$J_n(a, b, c, g; x_1, \dots, x_n) = \int_{C_1 \times \dots \times C_n} \prod_{i=1}^n t_i^a \prod_{i=1}^n (t_i - 1)^b \prod_{i=1}^n (t_i - x_i)^c \prod_{i < j} (t_i - t_j)^g dt_1 \dots dt_n \quad (1.3)$$

is studied, where the integration paths C_i satisfy the same conditions. Fuchsian systems (i.e. Pfaff systems of Fuchsian type) are obtained, which admit certain vector-valued functions associated to I_n , J_2 as solutions. For J_2 , two Fuchsian systems are given, one of them being equivalent to the Dotsenko–Fateev third order equation. The derivation of the Fuchsian systems is based on the theory of hyperplane arrangements, which provides some tools for handling integrals of hypergeometric type, whose integrands are products of complex powers of linear polynomials. The Fuchsian systems are important for applications, because they permit to obtain local expansions of solutions in the neighborhoods of singularities of the system [10]. Besides, a system of partial differential equations for J_n is obtained. This system is analogous to the Appell and Kampé de Fériet equations for hypergeometric functions of two complex variables F_1, F_2, F_3, F_4 .

The structure of the present paper is the following. In Sec. 2, a detailed description of the method for deriving Fuchsian systems for the vector-valued functions associated to hypergeometric integrals is given. This method consists in a description of a basis of sections of the cohomology bundle corresponding to the integral under consideration, followed by a computation of certain connection matrix in this basis. In Sec. 3, an explicit description of the basis is given. In Sec. 4, some auxiliary formulas, necessary for the computation of the connection matrix are obtained. In Sec. 5, 6, the cases of the integrals I_n, J_2 are considered. In Sec. 7, a system of partial differential equations for J_n is obtained. At last, in Sec. 8, another Fuchsian system for the Dotsenko–Fateev integral, equivalent to the Dotsenko–Fateev third order equation, is obtained by elementary methods.

2. Method for deriving Fuchsian systems

Let us consider the following hypergeometric integral:

$$I(x_1, \dots, x_n) = \int_{\sigma} \Phi dt, \quad \Phi = \prod_{i=1}^l \alpha_i(x_1, \dots, x_n; t_1, \dots, t_m)^{\lambda_i}, \quad dt = dt_1 \dots dt_m, \quad (2.1)$$

where $\alpha_i(x_1, \dots, x_n; t_1, \dots, t_m) = a_{i0}(x_1, \dots, x_n) + \sum_{j=1}^l a_{ij}(x_1, \dots, x_n)t_j$, $a_{ij}(x_1, \dots, x_n)$ are polynomials of variables x_1, \dots, x_n ; $\lambda_i \in \mathbb{C}$ are complex parameters with sufficiently large real parts, σ is the integration domain such that $\Phi|_{\partial\sigma} \equiv 0$, σ (σ may depend on the variables x_1, \dots, x_n).

Let $Z = \{(x, t) \in \mathbb{C}^n \times \mathbb{C}^m \mid \Phi(x, t) = 0\}$, and let $\text{pr} : Z \rightarrow \mathbb{C}^n$ be the corresponding projection. Then we have the family of hyperplane arrangements $Z_x = \text{pr}^{-1}(\{x\})$, $x \in \mathbb{C}^n$. We denote the complement of the arrangement Z_x in \mathbb{C}^m by $\mathbb{C}_x^m = \mathbb{C}^m \setminus Z_x$. For each $x \in \mathbb{C}^n$, we consider the operator $\nabla_x : \Omega^*(Z_x) \rightarrow \Omega^{*+1}(Z_x)$ defined by the following condition:

$$d(\Phi\psi) = \Phi\nabla_x\psi \quad \forall x \in \mathbb{C}^n \quad \forall \psi \in \Omega^*(Z_x),$$

where $\Omega^k(Z_x)$ is the vector space of rational differential k -forms on \mathbb{C}^m with poles along Z_x , d is the differential with respect to the variables t_1, \dots, t_m . It is easy to see that this operator has the form $\nabla_x = d + \omega_x \wedge$, where $\omega_x = \sum_{i=1}^l \lambda_i \omega_i \wedge$, $\omega_i = d \ln \alpha_i \in \Omega^1(Z_x)$. Besides, this operator is the differential for the following complex

$$0 \rightarrow \Omega^0(Z_x) \xrightarrow{\nabla_x} \Omega^1(Z_x) \xrightarrow{\nabla_x} \dots \xrightarrow{\nabla_x} \Omega^m(Z_x) \rightarrow 0. \quad (2.2)$$

The value of the integral $\int \Phi\psi$ at the point $x \in \mathbb{C}^n$ depends on the class of the differential form ψ in the highest cohomology $H^m(\Omega^*(Z_x), \nabla_x)$ by virtue of the following identities:

$$\int_{\sigma} \Phi \nabla_x \psi = \int_{\sigma} d(\Phi \psi) = \int_{\partial \sigma} \Phi \psi = 0 \quad (2.3)$$

whenever Φ decreases rapidly enough near $\partial \sigma$, so that the integrals converge.

Further consider the operator $\nabla' : \Omega^{0,\cdot}(*Z) \longrightarrow \Omega^{1,\cdot}(*Z)$ defined by the condition

$$d'(\Phi \psi) = \Phi \nabla' \psi \quad \forall \psi \in \Omega^{0,\cdot}(*Z),$$

where $\Omega^{p,q}(*Z)$ is the vector space of rational differential $(p+q)$ -forms on $\mathbb{C}^n \times \mathbb{C}^m$ with poles along Z , which are q -forms with respect to the variables t_1, \dots, t_m ; d' is the differential with respect to the variables x_1, \dots, x_n . Obviously, the operator ∇' has the expression $\nabla' = d' + \sum_{i=1}^l \lambda_i \omega'_i \wedge$, where $\omega'_i = d' \ln \alpha_i \in \Omega^{1,0}(*Z)$.

Now fix some point $x_0 \in \mathbb{C}^n$ and consider the subset $X \subset \mathbb{C}^n$ of all $x \in X$ such that the arrangements Z_x are combinatorially equivalent to Z_{x_0} (we restrict ourselves to an open set $X = \mathbb{C}^n \setminus \bigcup L_i$, where L_i are algebraic subvarieties of \mathbb{C}^n). Then $\mathcal{H} = \bigcup_{x \in X} H^m(\Omega^{0,\cdot}(*Z_x), \nabla_x)$ can be naturally endowed with a structure of a vector bundle over X , and the operator ∇' can be considered as a connection on this vector bundle. Suppose that the collection of logarithmic differential forms $\{\eta_i \in \Omega^{m,0}(*Z)\}$ defines the basis of global sections (i.e. it is a fiberwise basis) of the vector bundle \mathcal{H} . Then the action of the operator ∇' on the forms η_i is characterized by the formula

$$\nabla' \eta_i = \sum_{i=1}^n dx_i \wedge \left(\sum_j f_{ij}(x_1, \dots, x_n) \eta_j + \nabla(\psi_i) \right). \quad (2.4)$$

Upon integration over σ , we obtain

$$\begin{aligned} d' \left(\int_{\sigma} \Phi \eta_i \right) &= \int_{\sigma} d'(\Phi \eta_i) = \int_{\sigma} \Phi \nabla'(\eta_i) = \\ &= \sum_{i=1}^n dx_i \wedge \left(\sum_j f_{ij}(x_1, \dots, x_n) \int_{\sigma} \Phi \eta_j \right) = \sum_j \Omega_{ij} \wedge \int_{\sigma} \Phi \eta_j, \end{aligned} \quad (2.5)$$

where Ω_{ij} are differential 1-forms on X (in the case if σ doesn't depend on x , the first equality of (2.5) is evident; in the general case this equality follows from the condition $\Phi|_{\partial \sigma} \equiv 0$). Denote $\left(\int_{\sigma} \Phi \eta_i \right)^T$ by f , then the relations (2.5) have the form of the Pfaffian system $d'f = \Omega f$ on X . Moreover, if we choose all bounded chambers in $\mathbb{R}_x^m \subset \mathbb{C}_x^m$ as integration domains σ_j we obtain the basis of solutions $\left(\int_{\sigma_j} \Phi \eta_i \right)^T$ of this Fuchsian system [3].

Therefore, the problem of construction of Pfaffian system is reduced to the construction of a collection of rational logarithmic differential forms on $\mathbb{C}^n \times \mathbb{C}^m$ forming a basis of global sections of the vector bundle \mathcal{H} , and the computation of the connection matrix Ω in this basis. These problems are solved by methods of the theory of hyperplane arrangements which will be described in the next section. We will provide a collection of η_i such that the corresponding Pfaffian system is Fuchsian, the connection matrix Ω having the form $\sum_i A_i \frac{d' L_i}{L_i}$ with constant matrices A_i .

3. A basis of global sections

At first we describe a basis of the vector space $H^m(\Omega^{0,\cdot}(*\mathcal{A}), \nabla)$, where $\mathcal{A} = Z_{x_0}$, $\nabla = \nabla_{x_0}$, and then extend it to the desired collection of differential forms on X . Consider the graded subalgebra $B^{\bullet}(\mathcal{A})$ of

algebra $\Omega^*(\mathcal{A})$ generated by the differential forms $\omega_H = d \ln \alpha_H$ for all $H \in \mathcal{A}$. The inclusion $B^*(\mathcal{A}) \subset \Omega^*(\mathcal{A})$ induces an isomorphism of corresponding cohomologies $H^*(B^*(\mathcal{A}), \omega_\lambda) \simeq H^*(\Omega^*(\mathcal{A}), \nabla)$ (for a proof, see [7]).

Moreover the algebra $B^*(\mathcal{A})$ is isomorphic to the graded Orlik–Solomon algebra $A(\mathcal{A})$ of the arrangement \mathcal{A} . This algebra $A(\mathcal{A})$ is defined as the quotient of the Grassmannian algebra $\Lambda(\oplus_{H \in \mathcal{A}} \mathbb{C} e_H)$, where e_H are the formal variables corresponding to all hyperplanes of arrangement \mathcal{A} , by the ideal generated by the elements of the form $e_{H_{i_1}} \dots e_{H_{i_m}}$ for $H_{i_1} \cap \dots \cap H_{i_m} = \emptyset$ and the elements of the form $\partial(e_{H_{i_1}} \dots e_{H_{i_m}}) = \sum_{k=1}^m (-1)^{k-1} e_{H_{i_1}} \dots e_{H_{i_{k-1}}} e_{H_{i_{k+1}}} \dots e_{H_{i_m}}$ for $H_{i_1} \cap \dots \cap H_{i_m} \neq \emptyset$, $\text{codim}(H_{i_1} \cap \dots \cap H_{i_m}) < m$. This isomorphism $B^*(\mathcal{A}) \simeq A(\mathcal{A})$ is induced by the map $a_H \mapsto \omega_H$ where a_H is the projection of e_H in $A(\mathcal{A})$ (for a proof, see [9]).

Fix further a linear order on the set of hyperplanes from the arrangement \mathcal{A} and consider the set $2^{\mathcal{A}}$ of all subsets $S = \{H_{i_1}, \dots, H_{i_k}\}$ of the arrangements \mathcal{A} (final results do not depend on the choice of the order). We call a subset S dependent, if $\cap S = H_{i_1} \cap \dots \cap H_{i_m} \neq \emptyset$, $\text{codim}(\cap S) < |S|$, and independent, if $\cap S \neq \emptyset$, $\text{codim}(\cap S) = |S|$. A dependent subset of $2^{\mathcal{A}}$, minimal by inclusion, is called a circuit. Also we call S a broken circuit, if there exists a hyperplane $H \in \mathcal{A}$, such that $H < \min(S)$ and $S \cup H$ is a circuit. Now consider the collection $\text{nbc}(\mathcal{A})$ of subsets $S \in 2^{\mathcal{A}}$, such that $\cap S \neq \emptyset$ and S contains no broken circuits. It is easy to see that $\text{nbc}(\mathcal{A})$ is closed with respect to taking subsets of S . Hence, $\text{nbc}(\mathcal{A})$ is an abstract simplicial complex (the dimension of a simplex S is equal to $|S| - 1$). Further we will suppose that the dimension of $\text{nbc}(\mathcal{A})$ is equal to $m - 1$.

Now we consider cochain complex $C^*(\text{nbc}(\mathcal{A}), \mathbb{C})$ of the simplicial complex $\text{nbc}(\mathcal{A})$ with coefficients in \mathbb{C} . The map $\Theta : C^{*-1}(\text{nbc}(\mathcal{A}), \mathbb{C}) \rightarrow A^*(\mathcal{A})$ defined by the formula

$$\Theta(\alpha) = \sum_{S \in \text{nbc}(\mathcal{A}), |S|=q} \alpha(S) \bigwedge_{p=1}^q \left(\sum_{\cap_{k=p}^q H_{i_k} \subset H} \lambda_H a_H \right) \quad (3.1)$$

induces an isomorphism of cochain complexes $C^{*-1}(\text{nbc}(\mathcal{A}), \mathbb{C}) \simeq (A^*(\mathcal{A}), a_\lambda \wedge)$, where $a_\lambda \in A^*(\mathcal{A})$ correspond to the differential forms $\omega_\lambda \in B^*(\mathcal{A})$ (for a proof, see [3]). By this reason we have the following sequence of isomorphisms

$$H^q(\Omega^*(\mathcal{A}), \nabla) \simeq H^q(B^*(\mathcal{A}), \omega_\lambda) \simeq H^q(A^*(\mathcal{A}), a_\lambda \wedge) \simeq H^{q-1}(\text{nbc}(\mathcal{A}), \mathbb{C}). \quad (3.2)$$

We can obtain a basis of the highest cohomology $H^{m-1}(\text{nbc}(\mathcal{A}), \mathbb{C})$ of the simplicial complex $\text{nbc}(\mathcal{A})$ in the following way. Choose from the highest dimension simplices of $\text{nbc}(\mathcal{A})$ a subset $\beta\text{nbc}(\mathcal{A})$ of simplices S satisfying the following condition: for all hyperplanes $H \in S$, there exists a hyperplane $H' \in \mathcal{A}$, such that $(S \setminus H) \cup H'$ is a maximal independent subset of $2^{\mathcal{A}}$. Then the dual elements of $H^{m-1}(\text{nbc}(\mathcal{A}), \mathbb{C})$ corresponding to $\beta\text{nbc}(\mathcal{A})$ form a basis (for a proof, see [8]).

Now, using (3.2), we obtain a basis of $H^m(\Omega^*(\mathcal{A}), \nabla)$ defined as follows:

$$\theta(S) = \bigwedge_{p=1}^r \left(\sum_{\cap_{k=p}^r H_{i_k} \subset H} \lambda_H d \ln \alpha_H \right), \quad S \in \beta\text{nbc}(\mathcal{A}). \quad (3.3)$$

Applying this construction to our relative situation, where the hyperplanes of $\mathcal{A} = Z_x$ depend on x , we obtain a collection of differential forms on X whose cohomology classes form a basis of global sections of the vector bundle \mathcal{H} . Such a basis of global sections in the next sections will be called a βnbc -basis.

4. The connection matrix in a βnbc -basis

Let T be the $(m+1) \times (l+1)$ matrix, whose first $l = |\mathcal{A}|$ columns are filled with the coefficients of linear polynomials α_{H_i} , $i = 1, \dots, |\mathcal{A}| = l$ (the last entries of these columns being constant terms of the linear polynomials) and the last column is $(0, 0, \dots, 1)^T$. For an ordered collection of indices $S = (i_1, \dots, i_r)$,

we denote the collection $(i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_r)$ by S_k , the collection (i_1, \dots, i_r, j) by (S, j) and we use the following notation:

$$\alpha_S = \alpha_{i_1} \dots \alpha_{i_r}, \quad \omega_S = \omega_{i_1} \wedge \dots \wedge \omega_{i_r}, \quad \omega'_S = \sum_{k=1}^r (-1)^{k+1} \omega'_{i_k} \wedge \omega_{S_k}, \quad \Delta_S = \det T_S, \quad (4.2)$$

where T_S is the submatrix of T with columns indexed by S . If $r = m + 1$, it is easy to see that $\omega_{S_k} = \frac{\alpha_{i_k}}{\alpha_S} \Delta_{(S_k, n+1)} dt$ and the following equalities are true:

$$\begin{aligned} \omega'_S &= \sum_{k=1}^{m+1} (-1)^{k+1} \omega'_{i_k} \wedge \omega_{S_k} = \frac{1}{\alpha_S} \sum_{k=1}^{m+1} (-1)^{k+1} \Delta_{(T_k, n+1)} d_x \alpha_{i_k} \wedge dt = \frac{1}{\alpha_S} \sum_{k=1}^{m+1} (-1)^{k+1} d_x (\alpha_{i_k} \Delta_{(T_k, n+1)}) \wedge dt - \\ &\quad \frac{1}{\alpha_S} \sum_{k=1}^{l+1} (-1)^{k+1} \alpha_{i_k} d_x \Delta_{(T_k, n+1)} \wedge dt = \frac{1}{\alpha_S} d_x \left(\sum_{k=1}^{m+1} (-1)^{k+1} \alpha_{i_k} \Delta_{(T_k, n+1)} \right) \wedge dt - \\ &\quad \sum_{k=1}^{m+1} (-1)^{k+1} d_x \ln \Delta_{(T_k, n+1)} \wedge \frac{\alpha_{i_k}}{\alpha_S} \Delta_{(T_k, n+1)} dt = \frac{1}{\alpha_S} d_x \Delta_S \wedge dt - \sum_{k=1}^{m+1} (-1)^{k+1} d_x \ln \Delta_{(T_k, n+1)} \wedge \omega_{S_k} = \\ &\quad d_x \ln \Delta_S \wedge \frac{\Delta_S}{\alpha_S} dt - \sum_{k=1}^{m+1} (-1)^{k+1} d_x \ln \Delta_{(T_k, n+1)} \wedge \omega_{S_k} = \\ &\quad d_x \ln \Delta_S \wedge \left(\sum_{k=1}^{m+1} (-1)^{k+1} \frac{\alpha_{i_k}}{\alpha_S} \Delta_{S_k} dt \right) - \sum_{k=1}^{m+1} (-1)^{k+1} d_x \ln \Delta_{(T_k, n+1)} \wedge \omega_{S_k} = \\ &\quad \sum_{k=1}^{m+1} (-1)^{k+1} d_x \ln \Delta_S \wedge \omega_{S_k} - \sum_{k=1}^{m+1} (-1)^{k+1} d_x \ln \Delta_{(S_k, n+1)} \wedge \omega_{S_k} = \sum_{k=1}^{m+1} (-1)^{k+1} d_x \ln \frac{\Delta_S}{\Delta_{(S_k, n+1)}} \wedge \omega_{S_k}. \end{aligned}$$

On the other hand, for arbitrary r the following equalities also hold:

$$\begin{aligned} \nabla' \omega_S + \nabla \omega'_S &= \sum_{k=1}^r (-1)^{k+1} d_x \omega_{i_k} \wedge \omega_{S_k} + \sum_{i=1}^n \lambda_i \omega'_i \wedge \omega_S + \sum_{k=1}^r (-1)^{k+1} d_t \omega'_{i_k} \wedge \omega_{S_k} - \sum_{k=1}^r \lambda_{i_k} \omega'_{i_k} \wedge \omega_S + \\ &\quad \sum_{j \notin S} \lambda_j \sum_{k=1}^r (-1)^{r+k} \omega'_{i_k} \wedge \omega_{(S_k, j)} = \sum_{k=1}^r (-1)^{k+1} (d_x d_t + d_t d_x) \ln \alpha_{i_k} \wedge \omega_{S_k} + \left(\sum_{j \notin S} \lambda_j \omega'_j \right) \wedge \omega_T + \\ &\quad \sum_{j \notin S} \lambda_j \sum_{k=1}^r (-1)^{r+k} \omega'_{i_k} \wedge \omega_{(S_k, j)} = \left(\sum_{j \notin S} \lambda_j \omega'_j \right) \wedge \omega_S + \sum_{j \notin S} \lambda_j \sum_{k=1}^r (-1)^{r+k} \omega'_{i_k} \wedge \omega_{(S_k, j)} = (-1)^r \sum_{j \notin S} \lambda_j \omega'_{(S, j)}. \end{aligned}$$

Hence, in the highest cohomology $H^m(\Omega(*\mathcal{A}), \nabla)$ we have

$$\nabla' \omega_S \sim (-1)^{l+1} \sum_{j \notin S} \lambda_j \sum_{k=1}^{l+1} (-1)^k d_x \ln \frac{\Delta_{(S, j)}}{\Delta_{((S, j)_k, n+1)}} \wedge \omega_{(S, j)_k}. \quad (4.3)$$

Now, using (4.3), we can represent $\nabla' \theta(S)$ as a linear combination (with rational differential forms on \mathbb{C}^n as coefficients) of the sections ω_S of the vector bundle \mathcal{H} . Further it is necessary to expand the sections ω_S in the β nc-basis, but a general formula for this purpose is unknown. Below the corresponding computations for the integrals J_2, I_n are given.

5. Fuchsian system for $J_2(a, b, c, g; x, y)$

In this section we introduce the notation $x = x_1, y = x_2, t = t_1, s = t_2$. The integrand in the integral $J_2(a, b, c, g; x, y)$ at the point (x_0, y_0) such that $x_0 \neq 0, 1; y_0 \neq 0, 1; x_0 \neq y_0$ defines the following hyperplane arrangement:

$$\mathcal{J}_2 = \left\{ \{t = 0\}, \{t - 1 = 0\}, \{t - x_0 = 0\}, \{s = 0\}, \{s - 1 = 0\}, \{s - y_0 = 0\}, \{t - s = 0\} \right\}. \quad (5.1)$$

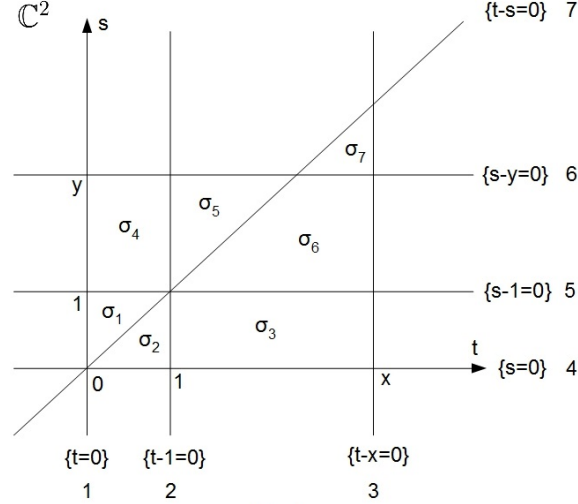


Fig. 1

We number the lines in this arrangement from 1 to 7 as shown on the figure. Then $\beta nbc(\mathcal{J}_2) = \left\{ \{3, 5\}, \{2, 6\}, \{3, 7\}, \{6, 7\}, \{3, 6\}, \{2, 7\}, \{2, 5\} \right\}$ is a βnbc -subset for the simplicial complex $nbc(\mathcal{J}_2)$. According to the formula (3.3) the corresponding collection of differential forms on $X = \mathbb{C}^2 \setminus \left(\{x = 0\} \cup \{x = 1\} \cup \{y = 0\} \cup \{y = 1\} \cup \{x = y\} \right)$, defining a βnbc -basis of the vector bundle \mathcal{H} , has the form

$$\begin{aligned} \eta_1 &= bc \frac{dt \wedge ds}{(t-x)(s-1)}, & \eta_2 &= bc \frac{dt \wedge ds}{(t-1)(s-y)}, \\ \eta_3 &= -cg \frac{dt \wedge ds}{(t-x)(t-s)}, & \eta_4 &= -cg \frac{dt \wedge ds}{(s-y)(t-s)}, & \eta_5 &= c^2 \frac{dt \wedge ds}{(t-x)(s-y)}, \\ \eta_6 &= -bg \frac{dt \wedge ds}{(t-1)(t-s)} - bg \frac{dt \wedge ds}{(s-1)(t-s)}, & \eta_7 &= b^2 \frac{dt \wedge ds}{(t-1)(s-1)} + bg \frac{dt \wedge ds}{(s-1)(t-s)}. \end{aligned} \quad (5.2)$$

Using the identities $\nabla_{(x,y)}(\alpha) \sim 0$ for the differential 1-forms $\alpha = d \ln(t-x), d \ln(s-y), d \ln(t-1), d \ln(s-1), d \ln t - d \ln s, d \ln(t-1) - d \ln(s-1), d \ln t, d \ln s$ one can obtain the following expansion of the sections ω_{ij} in the βnbc -basis

$$\begin{aligned} \omega_{34} &\sim -\frac{1}{ac}(\eta_1 + \eta_3 + \eta_5), & \omega_{16} &\sim -\frac{1}{ac}(\eta_2 - \eta_4 + \eta_5), & \omega_{24} &\sim -\frac{1}{ab}(\eta_2 + \eta_6 + \eta_7), \\ \omega_{15} &\sim -\frac{1}{ab}(\eta_1 + \eta_7), & \omega_{25} &\sim \frac{1}{b(2b+g)}(\eta_6 + 2\eta_7), & \omega_{27} &\sim \frac{1}{bg}(\eta_6 + \eta_7) - \frac{1}{g(2b+g)}(\eta_6 + 2\eta_7), \\ \omega_{14} &\sim \frac{1}{a(2a+g)}(2\eta_1 + 2\eta_2 + \eta_3 - \eta_4 + 2\eta_5 + \eta_6 + 2\eta_7), & \omega_{57} &\sim -\frac{1}{bg}\eta_7 + \frac{1}{g(2b+g)}(\eta_6 + 2\eta_7), \\ \omega_{17} &\sim -\frac{1}{g(2a+g)}(2\eta_1 + 2\eta_2 + \eta_3 - \eta_4 + 2\eta_5 + \eta_6 + 2\eta_7) + \frac{1}{ag}(\eta_1 + \eta_2 - \eta_4 + \eta_5 + \eta_7), \end{aligned}$$

$$\omega_{47} \sim \frac{1}{g(2a+g)}(2\eta_1 + 2\eta_2 + \eta_3 - \eta_4 + 2\eta_5 + \eta_6 + 2\eta_7) - \frac{1}{ag}(\eta_1 + \eta_2 + \eta_3 + \eta_5 + \eta_6 + \eta_7).$$

The corresponding matrix T , which is used in the formula (4.3), has the following form

$$T = \begin{pmatrix} 0 & -1 & -x & 0 & -1 & -y & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & 0 \end{pmatrix}.$$

Further using the formula (4.3) it is easy to compute the expansions of $\nabla' \eta_i$ in the β nbc-basis (5.2), and after taking the integrals we obtain the following proposition.

Proposition 1. The collection of the vector-valued functions $f_i = \left(\int_{\sigma_i} \Phi \eta_1, \dots, \int_{\sigma_i} \Phi \eta_7 \right)^T$, $i = 1, \dots, 7$ is the basis of solutions of the Fuchsian system

$$d'f = \left(A \frac{d'x}{x} + B \frac{d'y}{y} + C \frac{d'(x-1)}{x-1} + D \frac{d'(y-1)}{y-1} + E \frac{d'(x-y)}{x-y} \right) f, \text{ where}$$

$$A = \begin{pmatrix} a+c & 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ g & 0 & 2a+c+g & c & g & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & -c & a+c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a+c & 0 & 0 & 0 & c & c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -g & c & 2a+c+g & -g & c & 0 \\ c & 0 & c & 0 & a+c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} b+g & 0 & -b & 0 & 0 & 0 & -c \\ 0 & c & 0 & 0 & -b & 0 & 0 \\ -g & 0 & 2b & 0 & 0 & -c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c & 0 & 0 & b & 0 & 0 \\ g & 0 & -2b & 0 & 0 & c & 0 \\ -b-g & 0 & b & 0 & 0 & 0 & c \end{pmatrix}, \quad D = \begin{pmatrix} c & 0 & 0 & 0 & -b & 0 & 0 \\ 0 & b+g & 0 & b & 0 & -c & -c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & g & 0 & 2b & 0 & -c & 0 \\ -c & 0 & 0 & 0 & b & 0 & 0 \\ 0 & -g & 0 & -2b & 0 & c & 0 \\ 0 & -b & 0 & b & 0 & 0 & c \end{pmatrix},$$

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & -c & -g & 0 & 0 \\ 0 & 0 & -c & c & g & 0 & 0 \\ 0 & 0 & -c & c & g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

6. Fuchsian system for $I_n(a_0, a_1, \dots, a_n, g; x_2, \dots, x_n)$

In this section we introduce the notation $x_0 = x_0^0 = 0, x_1 = x_1^0 = 1$. The integrand in the integral $I_n(a_0, a_1, \dots, a_n, g; x_2, \dots, x_n)$ at the point $x_i^0 \neq x_j^0$, $0 \leq i \neq j \leq n$ defines the following hyperplane arrangement

$$\mathcal{I}_n = \left\{ V_i, H_i, D \mid i = 0, \dots, n \right\}, \quad V_i = \{t - x_i^0 = 0\}, \quad H_i = \{s - x_i^0 = 0\}, \quad D = \{t - s = 0\}. \quad (6.1)$$

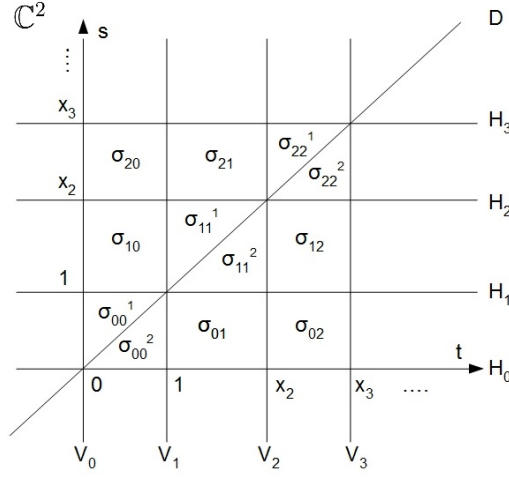


Fig. 2

It easy to see that the βnbc -subset of the simplicial complex $\text{nbc}(\mathcal{I}_n)$ is $\beta\text{nbc}(\mathcal{I}_n) = \left\{ \{V_i, H_j\} \mid 1 \leq i \leq n, 1 \leq j \leq n \right\} \cup \left\{ \{V_i, D\} \mid 1 \leq i \leq n \right\}$. Now using the formula (3.3) we compute the βnbc -basis of the vector bundle \mathcal{H} which is defined by the following differential forms

$$\theta(V_i, H_j) = a_i a_j V_i H_j, \quad \theta(V_i, H_i) = a_i^2 V_i H_i + a_i g D H_i, \quad \theta(V_i, D) = a_i g V_i D + a_i g H_i D, \quad 1 \leq i \neq j \leq n. \quad (6.2)$$

Here we use the notation $MN := d \log \alpha_M \wedge d \log \alpha_N$ for the hyperplanes M, N from the arrangement \mathcal{I}_n , where α_M, α_N are the linear polynomials defining the hyperplanes M, N .

The following identities are necessary in order to obtain the expansions of the sections MN corresponding to all pairs of hyperplanes from \mathcal{I}_n in the βnbc -basis (6.2):

$$\nabla_x(H_k) = \sum_{i=0}^n a_i V_i H_k + g D H_k = a_0 V_0 H_k + \sum_{i=1}^n \frac{1}{a_k} \theta(V_i, H_k) \sim 0, \quad 1 \leq k \leq n;$$

$$\nabla_x(V_k) = - \sum_{i=1}^n a_i V_k H_i - g V_k D = -a_0 V_k H_0 - \frac{1}{a_k} \left(\theta(V_k, H_k) + \theta(V_k, D) \right) - \sum_{i \neq k} \frac{1}{a_k} \theta(V_k, H_i) \sim 0;$$

$$\nabla_x(H_k - V_k) = \sum a_i V_i H_k - \sum a_i H_i V_k + g D(H_k - V_k) = (2a_k + g) V_k H_k + \sum_{i \neq k} a_i V_i H_k - \sum_{i \neq k} a_i H_i V_k =$$

$$(2a_k + g) V_k H_k + a_0 (V_0 H_k - H_0 V_k) + \sum_{i \neq 0, k} a_i (V_i H_k - H_i V_k) =$$

$$(2a_k + g) V_k H_k - \frac{1}{a_k} \left(\theta(V_k, D) + \sum_{i \neq 0} \theta(V_i, H_k) + \sum_{i \neq 0} \theta(V_k, H_i) \right) + \sum_{i \neq 0, k} \frac{1}{a_k} \left(\theta(V_i, H_k) + \theta(V_k, H_i) \right) \sim 0;$$

$$\nabla_x(H_0 - V_0) = \sum a_i V_i H_0 - \sum a_i H_i V_0 + g D(H_0 - V_0) = (2a_0 + g) V_0 H_0 + \sum_{i \neq 0} a_i (V_i H_0 - H_i V_0) =$$

$$(2a_0 + g) V_0 H_0 - \sum_{i \neq 0} \frac{1}{a_0} \left(\theta(V_i, D) + \sum_{j \neq 0} \theta(V_i, H_j) + \sum_{j \neq 0} \theta(V_j, H_i) \right) \sim 0;$$

$$\nabla_x(V_0) = a_0 H_0 V_0 + \sum_{i \neq 0} a_i H_i V_0 + g D V_0 =$$

$$\begin{aligned}
gDV_0 - \frac{1}{2a_0 + g} \left(\sum_{i \neq 0} \theta(V_i, D) + 2 \sum_{i,j \neq 0} \theta(V_i, H_j) \right) + \frac{1}{a_0} \sum_{i,j \neq 0} \theta(V_j, H_i) &\sim 0; \\
\nabla_x(H_0) = \sum a_i V_i H_0 + gDH_0 &= \\
gDH_0 + \frac{1}{2a_0 + g} \left(\sum_{i \neq 0} \theta(V_i, D) + 2 \sum_{i,j \neq 0} \theta(V_i, H_j) \right) - \frac{1}{a_0} \sum_{i \neq 0} \left(\theta(V_i, D) + \sum_{j \neq 0} \theta(V_i, H_j) \right) &\sim 0; \\
\theta(V_k, H_k) = a_k^2 V_k H_k + a_k gDH_k = a_k gDH_k + \frac{a_k}{2a_k + g} \left(\theta(V_k, D) + 2\theta(V_k, H_k) \right); \\
\theta(V_k, D) = a_k gV_k D + a_k gH_k D = a_k gV_k D + \frac{a_k}{2a_k + g} \left(\theta(V_k, D) + 2\theta(V_k, H_k) \right) - \theta(V_k, H_k).
\end{aligned}$$

The following expansions are straightforward consequences of the above identities:

$$\begin{aligned}
V_0 H_k &\sim -\frac{1}{a_0 a_k} \sum_{i \neq 0} \theta(V_i, H_k), \quad V_k H_0 \sim -\frac{1}{a_0 a_k} \left(\theta(V_k, D) + \sum_{i \neq 0} \theta(V_k, H_i) \right), \\
V_0 H_0 &\sim \frac{1}{(2a_0 + g)a_0} \left(\sum_{i \neq 0} \theta(V_i, D) + 2 \sum_{i,j \neq 0} \theta(V_i, H_j) \right), \quad V_k H_k \sim \frac{1}{(2a_k + g)a_k} \left(\theta(V_k, D) + 2\theta(V_k, H_k) \right), \\
V_0 D &\sim -\frac{1}{(2a_0 + g)g} \left(\sum_{i \neq 0} \theta(V_i, D) + 2 \sum_{i,j \neq 0} \theta(V_i, H_j) \right) + \frac{1}{a_0 g} \sum_{i,j \neq 0} \theta(V_i, H_j), \\
V_k D &\sim -\frac{1}{(2a_k + g)g} \left(\theta(V_k, D) + 2\theta(V_k, H_k) \right) + \frac{1}{a_k g} \left(\theta(V_k, D) + \theta(V_k, H_k) \right), \\
H_0 D &\sim \frac{1}{(2a_0 + g)g} \left(\sum_{i \neq 0} \theta(V_i, D) + 2 \sum_{i,j \neq 0} \theta(V_i, H_j) \right) - \frac{1}{a_0 g} \left(\sum_{i \neq 0} \theta(V_i, D) + \sum_{i,j \neq 0} \theta(V_i, H_j) \right); \\
H_k D &\sim \frac{1}{(2a_k + g)g} \left(\theta(V_k, D) + 2\theta(V_k, H_k) \right) - \frac{1}{a_k g} \theta(V_k, H_k), \quad 1 \leq k \leq n.
\end{aligned} \tag{6.3}$$

It is useful to note that all the determinants appearing in the formula (4.5) have the following simple form: $\Delta_{(V_i, H_j, V_k)} = x_i - x_k$, $\Delta_{(V_i, H_j, H_k)} = x_j - x_k$, $\Delta_{(V_i, H_j, D)} = x_i - x_j$, $\Delta_{(V_i, V_j, D)} = x_i - x_j$, $\Delta_{(H_i, H_j, D)} = x_i - x_j$. Further using the formula (4.5), one can compute:

$$\begin{aligned}
\nabla'(V_i H_j) &\sim \frac{d'(x_i - x_j)}{x_i - x_j} \wedge \left(a_j(H_j V_j + V_i H_j) + a_i(H_i V_i + V_i H_j) + g(H_j D - V_i D + V_i H_j) \right) + \\
&\sum_{k \neq j} \frac{d(x_i - x_k)}{x_i - x_k} \wedge a_k(H_j V_k + V_i H_j) + \sum_{k \neq i} \frac{d'(x_j - x_k)}{x_j - x_k} \wedge a_k(H_k V_i + V_i H_j), \\
\nabla'(V_i D) &\sim \sum_{k=0}^n \frac{d'(x_i - x_k)}{x_i - x_k} \wedge a_k(DV_k + DH_k + 2V_i D + H_k V_i), \\
\nabla'(H_i D) &\sim \sum_{k=0}^n \frac{d'(x_i - x_k)}{x_i - x_k} \wedge a_k(DV_k + DH_k + 2H_i D + V_k H_i).
\end{aligned} \tag{6.4}$$

Substituting (6.4) into (6.6) we obtain the expansions of $\nabla'(V_i H_j)$, $\nabla'(V_i D)$, $\nabla'(H_i D)$ in the βNBC -basis (6.2). Now we immediately obtain the connection matrix in the βNBC -basis. Therefore, after taking the integrals the following proposition follows.

Proposition 2. The collection of the vector-valued functions $f^{rsp} = \left(f_{ij}^{rsp}, f_k^{rsp}\right)_{i,j,k=1}^n$, where $f_{ij}^{rsp} = \int_{\sigma_{rs}^p} \theta(V_i, H_j)$, $f_k^{rsp} = \int_{\sigma_{rs}^p} \theta(V_k, D)$, $0 \leq r, s \leq n-1$ and $p = 1, 2$ for $r = s$, is the basis of solutions of the following Fuchsian system (the indices r, s, p are omitted):

$$\begin{aligned} d'f_{ij} &= \frac{d'(x_i - x_j)}{x_i - x_j} \wedge \left((a_i + a_j + g)f_{ij} - a_i f_{jj} - a_j f_{ii} - a_j f_i \right) + \frac{d'x_i}{x_i} \wedge \left(a_i \sum_{l \neq 0} f_{lj} + a_0 f_{ij} \right) + \\ &+ \sum_{k \neq 0, j} \frac{d'(x_i - x_k)}{x_i - x_k} \wedge \left(a_k f_{ij} - a_i f_{kj} \right) + \frac{d'x_j}{x_j} \wedge \left(a_j \left(f_i + \sum_{l \neq 0} f_{il} \right) + a_0 f_{ij} \right) + \sum_{k \neq 0, i} \frac{d'(x_j - x_k)}{x_j - x_k} \wedge \left(a_k f_{ij} - a_j f_{ik} \right), \\ d'f_{ii} &= \frac{d'x_i}{x_i} \wedge \left(\sum_{l \neq 0} \left((a_i + g)f_{li} + a_i(f_{il} - f_l) \right) + 2a_0 f_{ii} + a_i f_i \right) + \sum_{k \neq 0, i} \frac{d'(x_i - x_k)}{x_i - x_k} \wedge \left(a_i \left(f_k - f_{ik} - f_{ki} \right) + 2a_k f_{ii} - g f_{ki} \right), \\ d'f_i &= \frac{d'x_i}{x_i} \wedge \left((2a_0 + g)f_i + \sum_{l \neq 0} \left(2a_i f_l + g(f_{il} - f_{li}) \right) \right) + \sum_{k \neq 0, i} \frac{d'(x_i - x_k)}{x_i - x_k} \wedge \left(2a_k f_i - 2a_i f_k + g(f_{ki} - f_{ik}) \right). \end{aligned}$$

7. System of partial differential equations for $J_n(a, b, c, g; x_1, \dots, x_n)$

It is easy to see that the sections of the vector bundle \mathcal{H} defined by the differential forms $\prod_{i=1}^n (t_i - x_i)^{-k_i} dt$ correspond to the partial derivatives of the integral $J_n(a, b, c, g; x_1, \dots, x_n)$. To obtain a system of differential equations for the integral $J_n(a, b, c, g; x_1, \dots, x_n)$, one has to use the relations $\nabla_x(\beta_k) = 0$, where

$$\beta_k = \sum_{i=1}^n (-1)^{i+1} \frac{t_i(t_i - 1)}{t_i - x_i} \prod_{j \neq i} \left(1 + \frac{x_j - x_k}{t_j - x_j} \right) dt_1 \wedge \dots \wedge dt_{i-1} \wedge dt_{i+1} \wedge \dots \wedge dt_n, \quad k = 1, \dots, n. \quad (7.1)$$

The explicit form of these relations is given below:

$$\begin{aligned} 0 &= \sum_{i=1}^n \left(\frac{2t_i - 1}{t_i - x_i} - \frac{t_i(t_i - 1)}{(t_i - x_i)^2} \right) \prod_{j \neq i} \left(1 + \frac{x_j - x_k}{t_j - x_j} \right) dt + \\ &\sum_{i=1}^n \left(a \frac{t_i - 1}{t_i - x_i} + b \frac{t_i}{t_i - x_i} + c \frac{t_i(t_i - 1)}{(t_i - x_i)^2} \right) \prod_{j \neq i} \left(1 + \frac{x_j - x_k}{t_j - x_j} \right) dt + \\ &g \sum_{i < j} \frac{1}{\prod_{s=1}^n (t_s - x_s)} \frac{1}{t_i - t_j} \left(t_i(t_i - 1) \prod_{s \neq i} (t_s - x_k) - t_j(t_j - 1) \prod_{s \neq j} (t_s - x_k) \right) dt = \\ &\sum_{i=1}^n \prod_{j \neq i} \left(1 + \frac{x_j - x_k}{t_j - x_j} \right) \cdot \left(a \frac{t_i - 1}{t_i - x_i} + \frac{2t_i - 1}{t_i - x_i} + b \frac{t_i}{t_i - x_i} + (c - 1) \frac{t_i(t_i - 1)}{(t_i - x_i)^2} \right) dt + \\ &g \sum_{i < j} \frac{\prod_{s \neq i, j} (t_s - x_k)}{\prod_{s=1}^n (t_s - x_s)} \cdot \frac{1}{t_i - t_j} \cdot (t_i(t_i - 1)(t_j - x_k) - t_j(t_j - 1)(t_i - x_k)) dt. \end{aligned} \quad (7.2)$$

Now we apply some transformations:

$$\begin{aligned}
a \frac{t_i - 1}{t_i - x_i} &= a + a \frac{x_i - 1}{t_i - x_i}, \quad \frac{2t_i - 1}{t_i - x_i} = 2 + \frac{2x_i - 1}{t_i - x_i}, \quad b \frac{t_i}{t_i - x_i} = b + b \frac{x_i}{t_i - x_i}, \\
(c - 1) \frac{t_i(t_i - 1)}{(t_i - x_i)^2} &= (c - 1) \left(1 + \frac{2x_i - 1}{t_i - x_i} + \frac{x_i(x_i - 1)}{(t_i - x_i)^2} \right), \\
t_i(t_i - 1)(t_j - x_k) - t_j(t_j - 1)(t_j - x_k) &= t_i^2 t_j - t_i^2 x_k - t_i t_j - t_j^2 t_i + t_j^2 x_k + t_j t_i - t_j x_k = \\
&= t_i t_j (t_i - t_j) - (t_i - t_j)(t_i + t_j)x_k + (t_i - t_j)x_k = (t_i - t_j)(t_i t_j - (t_i + t_j - 1)x_k), \\
\frac{\prod_{s \neq i, j} (t_s - x_k)}{\prod_{s=1}^n (t_s - x_s)} &= \prod_{s \neq i, j} \left(1 + \frac{x_s - x_k}{t_s - x_s} \right) \cdot \frac{1}{(t_i - x_i)(t_j - x_j)}.
\end{aligned}$$

Plugging them in the formulas (7.2), we obtain:

$$\begin{aligned}
0 &= \sum_{i=1}^n \prod_{j \neq i} \left(1 + \frac{x_j - x_k}{t_j - x_j} \right) \cdot \left((1 + a + b + c) + \frac{(a + b + 2c)x_i - (a + c)}{t_i - x_i} + (c - 1) \frac{x_i(x_i - 1)}{(t_i - x_i)^2} \right) dt + \\
&\quad g \sum_{i < j} \prod_{s \neq i, j} \left(1 + \frac{x_s - x_k}{t_s - x_s} \right) \cdot \frac{t_i t_j - (t_i + t_j - 1)x_k}{(t_i - x_i)(t_j - x_j)} dt. \tag{7.3}
\end{aligned}$$

The last fraction of the identity (7.3) expands over the differential forms $\prod_{i=1}^n (t_i - x_i)^{-k_i} dt$ corresponding to the partial derivatives of our integral:

$$\begin{aligned}
\frac{t_i t_j - (t_i + t_j - 1)x_k}{(t_i - x_i)(t_j - x_j)} dt &= 1 + \frac{x_i}{t_i - x_i} + \frac{x_j}{t_j - x_j} + \frac{x_i x_j}{(t_i - x_i)(t_j - x_j)} - \frac{x_k}{t_i - x_i} - \frac{x_k}{t_j - x_j} - \frac{(x_i + x_j - 1)x_k}{(t_i - x_i)(t_j - x_j)} = \\
&= 1 + \frac{x_i - x_k}{t_i - x_i} + \frac{t_j - x_k}{t_j - x_j} + \frac{x_i x_j - (x_i + x_j - 1)x_k}{(t_i - x_i)(t_j - x_j)}.
\end{aligned}$$

Now after replacing of the differential forms by the corresponding partial derivatives we obtain the following proposition.

Proposition 3. The integral $J_n(a, b, c, g; x_1, \dots, x_n)$ is the solution of the system of n partial differential equations of the order $(n + 1)$:

$$P_k J_n = 0, \quad 1 \leq k \leq n, \quad \text{where}$$

$$\begin{aligned}
P_k &= \sum_{i=1}^n \prod_{j \neq i} (c - (x_j - x_k) \partial_j) \cdot ((1 + a + b + c)c - ((a + b + 2c)x_i - (a + c)) \partial_i + x_i(x_i - 1) \partial_i^2) + \\
&\quad g \sum_{i < j} \prod_{s \neq i, j} (c - (x_s - x_k) \partial_s) \cdot (c^2 - c(x_i - x_k) \partial_i - c(x_j - x_k) \partial_j + (x_i x_j - (x_i + x_j - 1)x_k) \partial_i \partial_j).
\end{aligned}$$

In particular, for $n = 2$ this system has the form $(x = x_1, y = x_2, u(x, y) = J_2(a, b, c, g; x, y))$:

$$\begin{cases} x(1-x)(x-y)u_{xxy} + cx(1-x)u_{xx} + cy(1-y)u_{yy} + \\ ((x-y)((a+b+2c)x - (a+c)) - gx(1-x))u_{xy} + \\ ((a+b+2c)x - (a+c))cu_x - ((x-y)(1+a+b+c+g) - \\ (a+b+2c)y + (a+c))cu_y - (2(1+a+b+c) + g)c^2u = 0, \\ y(1-y)(y-x)u_{xyy} + cx(1-x)u_{xx} + cy(1-y)u_{yy} + \\ ((y-x)((a+b+2c)y - (a+c)) - gy(1-y))u_{xy} + \\ ((a+b+2c)y - (a+c))cu_y - ((y-x)(1+a+b+c+g) - \\ (a+b+2c)x + (a+c))cu_x - (2(1+a+b+c) + g)c^2u = 0. \end{cases}$$

8. Another Fuchsian system for $I_2(a, b, c, g; z)$

It is known that the Dotsenko–Fateev integral $I_2(a, b, c, g; z)$ satisfies the following ordinary differential equation of third order [1, 2]:

$$y''' + \frac{K_1z + K_2(z-1)}{z(z-1)}y'' + \frac{L_1z^2 + L_2(z-1)^2 + L_3z(z-1)}{z^2(z-1)^2}y' + \frac{M_1z + M_2(z-1)}{z^2(z-1)^2}y = 0, \quad (8.1)$$

where coefficients have the form

$$\begin{aligned} K_1 &= -(3b + 3c + g), \quad K_2 = -(3a + 3c + g), \\ L_1 &= (b+c)(1+2b+2c+g), \quad L_2 = (a+c)(1+2a+2c+g), \\ L_3 &= (b+c)(1+2a+2c+g) + (a+c)(1+2b+2c+g) + (c-1)(a+b+c) + (3c+g)(1+a+b+c+g), \\ M_1 &= -c(2+2a+2b+2c+g)(1+2b+2c+g), \quad M_2 = -c(2+2a+2b+2c+g)(1+2a+2c+g). \end{aligned} \quad (8.2)$$

Denote $f_0 = (y, y', y'')^T$ and write (8.1) as the Pfaffian system $\frac{df_0}{dz} = \Omega_0 f_0$, where

$$\Omega_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{M_1}{z(z-1)^2} - \frac{M_2}{z^2(z-1)} & -\frac{L_1}{(z-1)^2} - \frac{L_2}{z^2} - \frac{L_3}{z(z-1)} & -\frac{K_1}{z-1} - \frac{K_2}{z} \end{pmatrix}. \quad (8.3)$$

Now transform this system, applying the multiplication of the vector-valued function f_0 by the following matrix:

$$\Gamma_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z-1 & 0 \\ 0 & 0 & (z-1)^2 \end{pmatrix}, \quad f_1 = \Gamma_0 f_0. \quad (8.4)$$

Then the system (8.3) takes the form $\frac{d}{dz}f_1 = \Omega_1 f_1$, where

$$\Omega_1 = \Gamma_0 \Omega_0 \Gamma_0^{-1} - \Gamma_0 \frac{d}{dz} \Gamma_0^{-1} = \begin{pmatrix} 0 & \frac{1}{z-1} & 0 \\ 0 & \frac{1}{z-1} & \frac{1}{z-1} \\ -\frac{M_1+M_2}{z} + \frac{M_2}{z^2} & -\frac{L_2+L_3}{z} - \frac{L_1}{z-1} + \frac{L_2}{z^2} & -\frac{K_2}{z} + \frac{2-K_1}{z-1} \end{pmatrix}. \quad (8.5)$$

The obstruction to being Fuchsian for the system (8.5) is the presence of summands of the form $\frac{C}{z^2}$ in the lower row of the matrix Ω_1 . To remove these summands, we transform the system (8.5), applying the multiplication of the vector-valued function f_1 by the following lower triangular matrix:

$$\Gamma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\eta}{z} & \frac{\zeta}{z} & 1 \end{pmatrix}, \quad f_2 = \Gamma_1 f_1. \quad (8.6)$$

The coefficients η, ζ are chosen to satisfy the two equations $M_2 + \eta\zeta + \eta K_2 - \eta = 0$, $L_2 + \zeta K_2 + \zeta^2 - \zeta = 0$, which provide the vanishing of the summands of the form $\frac{C}{z^2}$ in the lower row of the matrix Ω_2 of the transformed system (it is easy to see that this system has two solutions for $L_2 \neq 0$ and one solution for $L_2 = 0$). The resulting matrix Ω_2 has the following form:

$$\Omega_2 = \Gamma_1 \Omega_1 \Gamma_1^{-1} - \Gamma_1 \frac{d}{dz} \Gamma_1^{-1} = \quad (8.8)$$

$$= \begin{pmatrix} 0 & \frac{1}{z-1} & 0 \\ -\frac{M_1+M_2}{z} - \frac{\eta\zeta}{z-1} + \frac{\eta\zeta}{z} - \frac{(2-K_1)\eta}{z(z-1)} & \frac{\eta+\zeta}{z(z-1)} - \frac{L_2+L_3}{z} - \frac{L_1}{z-1} - \frac{\zeta^2}{z-1} + \frac{\zeta^2}{z} - \frac{(2-K_1)\zeta}{z(z-1)} & \frac{1}{z-1} - \frac{\zeta}{z(z-1)} + \frac{\zeta}{z} - \frac{K_2}{z} + \frac{2-K_1}{z-1} \end{pmatrix}$$

By collecting similar terms, we obtain the following proposition.

Proposition 4. Let y be a solution of the differential equation (8.1), and let the complex numbers η, ζ satisfy the two equations $M_2 + \eta\zeta + \eta K_2 - \eta = 0$, $L_2 + \zeta K_2 + \zeta^2 - \zeta = 0$. Then the vector-valued function $f = \Gamma_1 \Gamma_0 f_0 = (y, (z-1)y', \frac{\eta}{z}y + \zeta \frac{z-1}{z}y' + (z-1)^2 y'')^T$ is a solution of the Fuchsian system

$$df = \left(A \frac{dz}{z} + B \frac{dz}{z-1} \right) f, \text{ where}$$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ \eta & \zeta & 0 \\ -M_1 - M_2 + (\zeta + 2 - K_1)\eta & -\eta - L_2 - L_3 + \zeta^2 + (1 - K_1)\zeta & -\zeta - K_2 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 1 & 0 \\ -\eta & 1 - \zeta & 1 \\ (K_1 - \zeta - 2)\eta & \eta - L_1 - \zeta^2 - (1 - K_1)\zeta & 2 + \zeta - K_1 \end{pmatrix}.$$

Moreover, the map $y \mapsto f$ is an isomorphism between the space of solutions of (8.1) and the space of solutions of the Fuchsian system.

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